Malaysian Journal of Mathematical Sciences 11(2): 239–260 (2017)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Improved Robust Portfolio Optimization

Epha Diana Supandi*[1], Dedi Rosadi[2], and Abdurakhman[2]

[1]*Mathematics Study Program, State Islamic University, Indonesia*
[1]*Doctoral Student, Mathematics Department, Gadjah Mada University, Indonesia*
[2]*Mathematics Department, Gadjah Mada University, Indonesia*

E-mail:epha.supandi@uin-suka.ac.id
**Corresponding author*

ABSTRACT

A robust optimization has emerged as a powerful tool for managing uncertainty in many optimization problems. This method was adapted in portfolio optimization to resolve the sensitivity issue of the mean-variance model to its inputs (i.e. mean vector and covariance matrix of returns). The solution provided by this framework presented here can be very sensitive to the choice of uncertainty sets, since the optimal portfolios are determined under "the worst-case objective value" of the inputs in their uncertainty sets. One potential consequence of this emphasis on the worst-case is that the decisions are highly influenced by extreme scenarios in the uncertainty sets. The emergence of the extreme scenarios in the uncertainty sets can be because there are extreme observations in the data. These extreme observations frequently occur in financial sector. We proposed to tackle this issue by considering robust estimators that are incorporated to the uncertainty sets about unknown parameters. They showed both in simulated and empirical investigations that this strategy can lead to the construction of portfolios with superior out-of-sample performance in comparison to the mean-variance portfolio (classic) and robust portfolio optimization.

Keywords: Mean-variance portfolio, robust portfolio optimization, robust estimators, uncertainty sets, block bootstrap.

1. Introduction

Sixty four years ago, Markowitz (1952) developed a portfolio selection theory that became the foundation of financial economics for asset management and revolutionized investment practice. It is assumed that random vector $r = (r_1, r_2, \ldots\ldots, r_p)^T$ denotes random returns of the p risky assets with mean vector μ and covariance matrix Σ. A portfolio is defined to be a list of weights w_i for the assets $i = 1, \ldots, p$ that represent the amount of capital to be invested in each asset. We assumed that $\sum_{i=1}^{p} w_i = 1$ meaning that capital is fully invested.

For a given portfolio w , the expected return and variance are respectively given by: $E(w^T r) = w^T \mu$ and $Var(w^T r) = w^T \Sigma w$. Then, the classical mean-variance (MV) portfolio models of Markowitz are formulated mathematically as the optimization problem:

$$\begin{aligned} \max_{w} \quad & w^T \mu - \frac{\gamma}{2} w^T \Sigma w \\ s.t: \quad & w^T e = 1 \\ & w \geq 0 \end{aligned} \tag{1}$$

where $\mu \in \mathbb{R}^p$ is the vector of expected return, $\Sigma \in \mathbb{R}^{p \times p}$ is the covariance matrix of return, and $w \in \mathbb{R}^p$ is the vector of portfolio weight. Restriction $w > 0$ means that short-selling is not allowed. Parameter γ can be interpreted as a risk aversion, since it takes into account the trade-off between risk and return of the portfolios.

For the empirical implementation, the MV portfolio (classic) is the solution to optimization problem (1):

$$\begin{aligned} \max_{w} \quad & w^T \hat{\mu} - \frac{\gamma}{2} w^T \hat{\Sigma} w \\ s.t: \quad & w^T e = 1 \\ & w \geq 0 \end{aligned} \tag{2}$$

where $\hat{\mu}^T w$ is the sample mean of portfolio returns, $w^T \hat{\Sigma} w$ is the sample variance of portfolio returns. Suppose the number of return is n then the sample mean of returns can be calculated as $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} r_i$ and the sample covariance matrix of asset returns can be calculated as $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (r_i - \hat{\mu})(r_i - \hat{\mu})^T$. We could generate an optimal portfolio by choosing various values of risk aversion parameter.

According to Lauprete (2001), solving problem (2) can be expected optimal if the data are multivariate normal distributed. In actual financial market, the empirical distribution of asset returns may in fact asymmetry therefore model (2) are very erroneous (see Best and Grauer (1991); Broadie (1993); (Chopra and Ziemba, 1993); Ceria and Stubbs (2006)).If unchecked, this phenomenon skews the optimizer towards extreme weights that tend to perform poorly in the real world. One solution that has been proposed is to use robust optimization (Rob.Opt). This method has emerged as a powerful tool for managing uncertainty in many optimization problems. Robust optimization was developed to solve problems where there is uncertainty in the decision environment, and therefore is sometimes referred to as uncertain optimization (Ben-Tal and Nemirovski (1998)). This study was adapted in portfolio optimization to resolve the sensitivity issue of the MV model to its inputs. Goldfarb and Iyengar (2003) introduce a way to formulate robust optimization problems as second-order cone programs using ellipsoidal uncertainty sets, and similar approaches are investigated by Tütüncü and Koenig (2004), Garlappi et al. (2004), Fastrich and Winker (2009), Lu (2011) and Supandi et al. (2016) apply this idea in different ways. Meanwhile Fabozzi et al. (2007) provide a comprehensive overview for this framework.

In the Rob.Opt framework, input parameters are modeled as unknown, but belong to bound uncertainty sets that contain all, or most, values of uncertain inputs. Furthermore, robust optimization determines the optimal portfolio under "the worst-case objective value" of the inputs in their uncertainty sets (Fabozzi et al. (2007)). The focus on the worst-case objective value in robust optimization is a source of frequent criticism. Despite the advantages we mentioned in the previous paragraph, Rob.Opt provides a conservative framework to determine an optimal portfolio under model parameter uncertainty. As stated by Zhu (2008) such a framework tends to be too pessimistic and unable to achieve high portfolio returns, especially for less risk-averse investors. In addition, the solution provided by this framework can be very sensitive to the choice of uncertainty sets. One potential consequence of this emphasis on the worst-case is that the decisions are highly influenced by extreme scenarios in the uncertainty sets. As this is not always desirable, Bertsimas and Sim (2004) study this cost of robustness as a function of the level of conservatism.

The emergence of the extreme scenarios in the uncertainty sets can be because there are extreme observations in the data. These extreme observations frequently occur in the financial sector; such observations are called outliers or sometimes referred to as contaminants. To reduce the influence of the outliers on the Rob.Opt portfolio, we proposed to use robust estimators in a construction of the uncertainty sets. A robust estimator is one that gives meaningful information about asset returns even when the empirical (sample) distribution deviates from the assumed (normal) distribution (see Marona et al. (2006), Huber and Ronchetti (2009)). The focus on robust estimation is to produce efficient estimators, which are less sensitive to outliers and might result in more robust portfolios. Hence, one could use a robust estimators generated using robust estimation to determine uncertainty sets, from which the worst-case parameters are chosen for robust optimization.

For this reason, this paper studies the effectiveness of robust optimization combined with robust estimation. In addition, it asks whether combining robust optimization with robust estimation can do better performance than previous techniques (Classic and Rob.Opt). We started by giving a formal description of the problem. Next, we briefly recalled the robust portfolio optimization of Tütüncü and Koenig (2004) and robust estimation i.e. S-Estimators of Davies (1987). Thereafter, we described how to integrate robust estimation to robust portfolio optimization. Finally, we computed the out-of-sample performance of the proposed method by extensive simulated and empirical investigations on various data sets.

2. Robust Portfolio Optimization

Many decision problems with uncertainty can be formulated as optimization problems. In recent years, robust optimization (Rob.Opt) has emerged as a powerful tool for managing uncertainty in such optimization problems (Ben-Tal and Nemirovski (1998), Ben-Tal and Nemirovski (2002)).

This method was adapted in portfolio optimization to resolve the sensitivity issue of the MV portfolio to its inputs. One of the essential elements of Rob.Opt model is the uncertainty sets. The uncertainty sets, say $\mathcal{U}$, represent the set of possible scenarios or realizations for parameters μ and Σ. When parameters are uncertain and must be estimated, uncertainty sets can represent or be formed by differences of opinions (see Fabozzi et al. (2007)). For example, they could be discrete sets representing a collection of estimates for the unknown input parameters. In relation to this, Goldfarb and Iyengar (2003) consider ellipsoidal uncertainty sets, while, Tütüncü and Koenig (2004) assume that uncertainty mean return vector μ and uncertainty covariance matrix Σ of the asset return belong to the uncertain sets of the following forms:

$$\mathcal{U}_\mu = \{\mu : \mu^L \leq \mu \leq \mu^U\} \tag{3}$$

and

$$\mathcal{U}_\Sigma = \{\Sigma : \Sigma^L \leq \Sigma \leq \Sigma^U, \Sigma \succeq 0\} \tag{4}$$

Here, μ^L and μ^U are the lower bound and upper bound of mean return respectively, whereas Σ^L and Σ^U denote the lower bound and upper bound of covariance matrix. The restriction $\Sigma \succeq 0$ indicates that Σ is a symmetric positive semidefinite matrix.

Given the uncertainty set $\mathcal{U}_\mu$ and $\mathcal{U}_\Sigma$, the robust versions of the MV portfolio (2) can be expressed as follows

$$\begin{aligned}
\max_{w} \quad & \min_{\mu \in \mathcal{U}_\mu} [\mu^T w] - \frac{\gamma}{2} \max_{\Sigma \in \mathcal{U}_\Sigma} [w^T \Sigma w] \\
s.t: \quad & w^T e = 1 \\
& w \geq 0 \\
& \mu \in \mathcal{U}_\mu \\
& \Sigma \in \mathcal{U}_\Sigma
\end{aligned} \tag{5}$$

The robust portfolio optimization can be determined by first fixing the worst-case input data in the considered uncertainty sets. Since $w > 0$, the objective value of the problem in (5) is minimized when each element of vector μ is at its lower bound, i.e., when $\mu = \mu^L$. Consider the relaxation of the second problem obtained by ignoring the positive semi-definiteness constraint $\Sigma \succeq 0$. For this relaxation, since $w_i w_j > 0$ for all i and j, $w^T \Sigma w = w_{i,j} \sigma_{ij} w_i w_j$ will be maximized when all $\sigma_i j$ attain their largest feasible values, i.e., when $\Sigma = \Sigma^U$. Since Σ^U is assumed to be a positive semidefinite matrix, it must be optimal for the unrelaxed problem as well (see for detail in Tütüncü and Koenig (2004).

In this scenario, the robust portfolio optimization given in (5) reduces to the following maximization problem:

$$\begin{aligned}
\max_{w} \quad & (\mu^L)^T w - \frac{\gamma}{2} \left[w^T \Sigma^U w \right] \\
s.t: \quad & w^T e = 1 \\
& w \geq 0
\end{aligned} \tag{6}$$

3. Improved Robust Portfolio Optimization

We have seen in the previous section how to write the robust portfolio optimization (Rob.Opt) problem. In this formulation, we needed only a lower bound of $\mathcal{U}_\mu$ and upper bound of $\mathcal{U}_\Sigma$ which are entered into the MV model (see formula [4]). Since the worst-case based approach can be adversely influenced by outliers in the data, uncertainty sets need to be carefully chosen. This is why we proposed to include robust estimators in construction of Rob.Opt model. Robust estimators are one of the most effective approaches to mitigate the impact of outliers in data set.

We enriched the robust portfolio optimization model of Tütüncü and Koenig (2004) from two perspectives. First, we developed block bootstrap to determine the uncertainty sets of parameters (see Efron and Tibshirani (1993)). Second, as stated by Zhu (2008) this framework tends to be too pessimistic and unable to achieve high portfolio returns therefore we introduced robust estimators for reducing the weakness of Rob.Opt model. Spesifically we developed the robust portfolio optimization model of Tütüncü and Koenig (2004) with S-Estimators as proposed by Davies (1987).

One of the robust estimators for location and scale with multivariate data is an S-Estimators. According to Rocke (1996), an S-estimate of multivariate location and shape is defined as μ vector and positive definite symmetric (PDS) matrix Σ which minimize objective function:

$$\min \quad |\Sigma| \tag{7}$$

$$s.t: \quad \frac{1}{n}\sum_{i=1}^{n}\rho(d_i) = b_0 \tag{8}$$

where $d_i = (r_i - \mu)^T\Sigma^{-1}(r_i - \mu)$ and ρ is the loss function and it should have the following properties:

1. ρ is symmetric, has a continuous derivative ψ and $\rho(0) = 0$
2. There exists a finite constant $c_0 > 0$ such that ρ is strictly increasing on $[0, c_0]$ and constant on $[c_0, \infty]$

Constant b_0 is generally chosen to be $b_0 = \epsilon\rho(c_0)$ for breakdown ϵ. Often a value of ϵ near 0.5 is used to obtain very high breakdown.

Let $\psi(d) = \rho'(d)$, $u(d) = \psi(d)/d$ and $d_i = (r_i - \mu)^T\Sigma^{-1}(r_i - \mu)$ then an S-estimate $\hat{\theta} = (\hat{\mu}, \hat{\Sigma})$ satisfies the following estimating equations (Lopuha, 1989):

$$1/n\sum_{i=1}^{n}u(d_i)(r_i - \hat{\mu}) = 0 \tag{9}$$

$$1/n\sum_{i=1}^{n}pu(d_i)(r_i - \hat{\mu})(r_i - \hat{\mu})^T - \nu(d_i)\hat{\Sigma} = 0 \tag{10}$$

where $\upsilon(d) = \psi(d)d - \rho(d) + b_0$. More details about S-Estimators can be found in the references, for instance Davies (1987), Lopuhaa (2009) and Rocke (1996). The main difference between our proposed model and those proposed by Tütüncü and Koenig (2004) is that we used the S-Estimators as the estimation in the construction of uncertainty sets of parameters. We give the detail of the procedure in the following algorithm.

Algorithm 1. Uncertainty Sets of Parameters by Using a Block Bootstrap Methods

1. Choose the block length (l). In our experiment, we used the non - overrlapping block. We divided the data into n/l blocks, in which block 1 is $r_1, r_2, \ldots, r_l$, block 2 is $r_{l+1}, r_{l+2}, \ldots, r_{2l}, \ldots$, etc,
2. Resample the blocks and generate the bootstrap sample
3. Compute the S estimators (Equation 9) from bootstrap data and call it $\hat{\mu}_{Sest}$ and $\hat{\Sigma}_{Sest}$,
4. Construct the empirical distribution of estimators by repeating step 2 and step 3 B times and sort the bootstrap estimators from the smallest to largest,
5. Determine the $(1-\alpha)100\%$ percent quantile of the distribution of estimators.

From algorithm 1, the uncertainty sets are defined as

$$\mathcal{U}_\mu = \left\{\mu : \hat{\mu}^L_{Sest} \le \mu \le \hat{\mu}^U_{Sest}\right\} \tag{11}$$

And

$$\mathcal{U}_\Sigma = \left\{\Sigma : \hat{\Sigma}^L_{Sest} \le \Sigma \le^U_{Sest}, \Sigma \succeq 0\right\} \tag{12}$$

Given the uncertainty sets of mean vector (11) and covariance matrix (12, then our proposed modified robust optimization (Mod.Rob) can be defined as follows:

$$\begin{aligned} \max_w \quad & (\hat{\mu}^L_{sest})^T w - \frac{\gamma}{2}\left[w^T \hat{\Sigma}^U_{sest} w\right] \\ s.t: \quad & w^T e = 1 \\ & w \ge 0 \end{aligned} \tag{13}$$

4. Simulation Study

We considered that the returns follow a multivariate normal distribution most of the time but there is a small probability that the returns follow a different distribution. That is, we assumed that the true asset-return distribution is:

$$M = (1-\varepsilon)N_p(\mu,\Sigma) + \varepsilon D \tag{14}$$

Here, M can be considered as a mixed distribution between a multivariate normal distribution in dimension $p : N_p(\mu,\Sigma)$ and contamination distribution D. Whereas $\epsilon \in (0,1)$ is a number representing the proportion of contamination. Furthermore, we would consider a case where $D = N_p(\mu_d, \Sigma_d)$ meaning that D is also a multivariate normal distribution but with different parameters. Specifically, we assumed that $\mu_d = -\mu$ and $\Sigma_d = \Sigma$.

In order to demonstrate the effect of contamination on the computation of Rob.Opt portfolio, we conducted the following experiment. If there are ten risky assets and their true parameters, mean vector μ $(\times 10^{-3})$ is

$$\mu^T = \begin{pmatrix} 0.80 & 2.66 & 1.28 & 3.23 & 1.14 & 5.43 & 3.91 & 2.34 & 2.95 & 4.13 \end{pmatrix}$$

and covariance matrix Σ $(\times 10^{-3})$ is

$$\Sigma = \begin{pmatrix} 5.22 & 1.97 & 1.55 & 1.44 & 1.70 & 0.71 & 1.66 & 1.96 & 1.49 & 0.71 \\ & 6.52 & 2.16 & 2.19 & 3.28 & 1.11 & 1.98 & 2.05 & 2.22 & 1.42 \\ & & 4.27 & 2.47 & 2.14 & 0.76 & 1.34 & 1.37 & 1.85 & 0.74 \\ & & & 3.02 & 1.79 & 0.53 & 1.15 & 1.18 & 1.73 & 0.75 \\ & & & & 7.80 & 1.01 & 1.98 & 2.04 & 2.04 & 1.57 \\ & & & & & 4.86 & 0.87 & 0.82 & 0.71 & 0.18 \\ & & & & & & 3.24 & 2.58 & 1.48 & 0.53 \\ & & & & & & & 3.08 & 1.51 & 0.54 \\ & & & & & & & & 3.43 & 0.69 \\ & & & & & & & & & 4.2114 \end{pmatrix}$$

We generated two different data sets with the proportions of the data deviating from normality ε equal to 0% and 5%. This allows us to study how the different portfolios change when the asset-return distribution deviates from the normal distribution. We generated 200 return samples (we can consider the samples as 200 weekly returns of the ten assets). From these samples, we calculated the optimal robust portfolio by using Equation 2, 6 and 13.

We repeated the process with various types of risk aversion ($\gamma = 1$, 10, 100 and 1000). As the name indicates, it is a measure of the investors' risk averseness. It can be different for each investor, and even for an investor it can change through time. The greater the γ , the more risk averse the investor has. (Engels (2004))

Next, we used a rolling horizon procedure similar to that in DeMiguel et al. (2013) to compare the performance of our proposed model with Classic and Rob.Opt strategy. The detail of the method can be defined as follows:

Algorithm 2. Rolling Horizon Procedure

1. Choose the length of estimation window K where $K < n$ and n is a total number of sample in a data set;
2. Compute the optimal robust portfolio for each strategy, call it $w^*_{classic}$, $w^*_{rob.opt}$ and $w^*_{mod.rob}$;
3. Calculate the out-of-sample excess return in period $t+1$ i.e. $\hat{r}_{t+1} = (w^*_t)^T r_{t+1}$ where $K \leq t \leq 200$;
4. Repeat step 2 for the next window by including the next data point and drop the first data point of the estimation window (we assumed that investors would rebalance their portfolios every one week);
5. After collecting the time series of the excess returns $\hat{r}_{t+1}$, calculate the out-of-sample mean, standard deviation, Sharpe ratio of excess returns and turnover for each strategy.

To perform the out-of-sample performances, several values have to be set. Firstly, we chose an estimation window of $K = 90$ observations, since we generated $n = 200$ return samples then we had 110 observations for out-of-sample evaluation . Secondly, block bootstrap method was used by setting number of bootstrap $B = 500$ and length of bootstrap $l = 10$. We used algorithm 1 to compute the uncertainty sets by using block bootstrap percentil with $\alpha = 0.10$. Thirdly, because we incorporated robust estimators (i.e. S-Estimators) in the construction of uncertainty sets of parameters, we had to determine what type of loss function we would use. In this case, we computed S-Estimators based on Tukey's biweight function.

4.1 Performance with Simulated Data

Table 1 reports the results of MV portfolio (Classic), robust optimization portfolio (Rob.Opt) and modified robust optimization portfolio (Mod.Rob) at different risk aversions when applied to the simulated data. The first column of Table 1 displays the criteria of performances i.e. mean, standard deviation (Std Dev), Sharpe Ratio (SR) and turnover (TO). The second, third and fourth columns show the performance of three portfolios in uncontaminated data (no outlier). Meanwhile, the last three columns display the performance of Classic, Rob. Opt and Mod.Rob portfolio for contaminated data ($\varepsilon = 0.05$).

Table 1: Out-of-sample Mean, Standard Deviation, Sharpe Ratio and Turnover in in Simulated Data Set

	($\varepsilon = 0\%$)			($\varepsilon = 5\%$)		
	Classic	Rob.Opt	Mod.Rob	Classic	Rob.Opt	Mod.Rob
$\gamma = 1$						
Mean	0.0083	0.0082	0.0074	-0.0109	-0.0111	-0.0032
St.Dev	0.0616	0.0757	0.0740	0.0860	0.0814	0.0775
SR	0.1355	0.1084	0.1005	-0.1267	-0.1363	-0.0412
TO	1.9026	1.9472	1.9877	1.4971	1.5372	1.5114
$\gamma = 10$						
Mean	0.0011	0.0005	0.0010	0.0073	0.0087	0.0091
St.Dev	0.0428	0.0448	0.0466	0.0512	0.0505	0.0511
SR	0.0259	0.0108	0.0226	0.1434	0.1716	0.1781
TO	1.0268	1.0644	1.1667	1.7832	1.5545	1.4054
$\gamma = 100$						
Mean	0.0021	0.0018	0.0019	0.0067	0.0071	0.0074
St.Dev	0.0374	0.0376	0.0375	0.0437	0.0436	0.0445
SR	0.0562	0.0478	0.0506	0.1532	0.1631	0.1662
TO	1.1379	1.2442	1.2671	1.1836	1.1374	1.1834
$\gamma = 1000$						
Mean	0.0049	0.0043	0.0041	0.0111	0.0109	0.0112
St.Dev	0.0365	0.0373	0.0380	0.0395	0.0396	0.0390
SR	0.1353	0.1165	0.1070	0.2811	0.2762	0.2877
TO	1.0179	1.1271	1.1728	1.1177	1.1124	1.0900

First, we analyzed out-of-sample performances when returns follow multivariate normal distribution (uncontaminated data). We could check that the all of out-of-sample performances (mean, standar deviation, Sharpe ratio and turnover) of the classical portfolios are outperformed than those obtained with robust portfolio policies. It is not surprising, because classical portfolios are based on the sample mean vector and sample covariance matrix, which are the maximum likelihood estimators (MLE) for normally distributed returns.

For instance, in case $\gamma = 1$, the Sharpe ratio of the classic (MV) portfolio is 0.1355 whereas the Sharpe ratio for Rob.Opt and Mod.Rob portfolio is 0.1084 and 0.1005, respectively. Moreover, the portfolio turnover of the classic method is smaller than those of robust approaches in the majority of the specifications. For instance, in the case $= 1$ the portfolio turnover of the classic portfolio was is 1.9026 whereas the portfolio turnovers of the Rob.Opt and Mod.Rob porfolios are 1.9472 and 1.9877, respectively.

The average turnover measures the rate of trading activity across portfolio assets. As such, it represents the percentage of portfolio that is bought and sold in exchange for other assets. Clearly, the smaller the turnover, the smaller the transaction costs associated to the implementation of the strategy.

Next, we examined the out-of-sample performances on the contaminated data (the proportions of outlier ε is equal to 5%). According to the out-of-sample performances (i.e. mean, standard deviation, sharpe ratio and turnover), we observed that Mod.Rob achieves higher out-of-sample means and Sharpe ratio than Classic and Rob.Opt. Furthermore, the out-of-sample turnovers of Mod.Rob are lower for all risk aversion scenarios compared to those of Classic and Rob.Opt. In addition, portfolios generated using Mod.Rob deliver lower risks for $= 1$ and 100. Overall, Mod.Rob portfolios lead to superior results compared to the Classic and Rob.Opt approaches in the contaminated data. On the contrary, under contamination, the Classical and Rob.Opt portfolios lead to the worst out-of-sample performance. This is particularly worrying in finance, where there is extensive evidence that the empirical return distributions often depart from normality.

The conclusion is that when the sample distribution deviates even slightly from the assumed distribution, the efficiency of classical estimators may drastically decrease. Robust estimators, on the other hand, are not as efficient as MLE when the underlying model is correct, but their properties are not too sensitive to deviations from the assumed distribution. Hence, the proposed modified robust portfolio optimization approach works effectively with simulated datasets where the available observations depart from the normality assumption. Moreover, it can be noticed for all portfolios (see Table 1), by comparing portfolio turnover values, that in most portfolios, increasing the risk aversion from 1 to 1000 has caused a decrease in these values.

To gain further meaningful insight, we also evaluated the stability of all portfolios through the performance of portfolio weights. We produced a graphical representation of the stability of different portfolio policies by using the boxplots of portfolio weights. Each boxplot represents the variability of the portfolio weight assigned to a particular asset by a particular policy. Clearly, stable policies should have relatively compact (short) boxplots.

Figure 1 exhibits the portfolio weights on the uncontaminated data, while Figure 2 shows the boxplot on the contaminated data. Each panel contains 10 boxplots corresponding to each of the ten assets. The ten assets are labeled as 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10. The boxplots for the portfolio weights for different values of γ are given per line, that is risk aversion parameter $\gamma = 1$ (the first row of the graphs), $\gamma = 10$ (the second row of the graphs), $\gamma = 100$ (the third row of the graphs) and $\gamma = 1000$ (the fourth row of the graphs).

Figure (1) and (2) confirm the analysis of the out-of-sample turnover for all methods, that greater risk aversion will produce lower turnover value, meaning that portfolio is more stable.

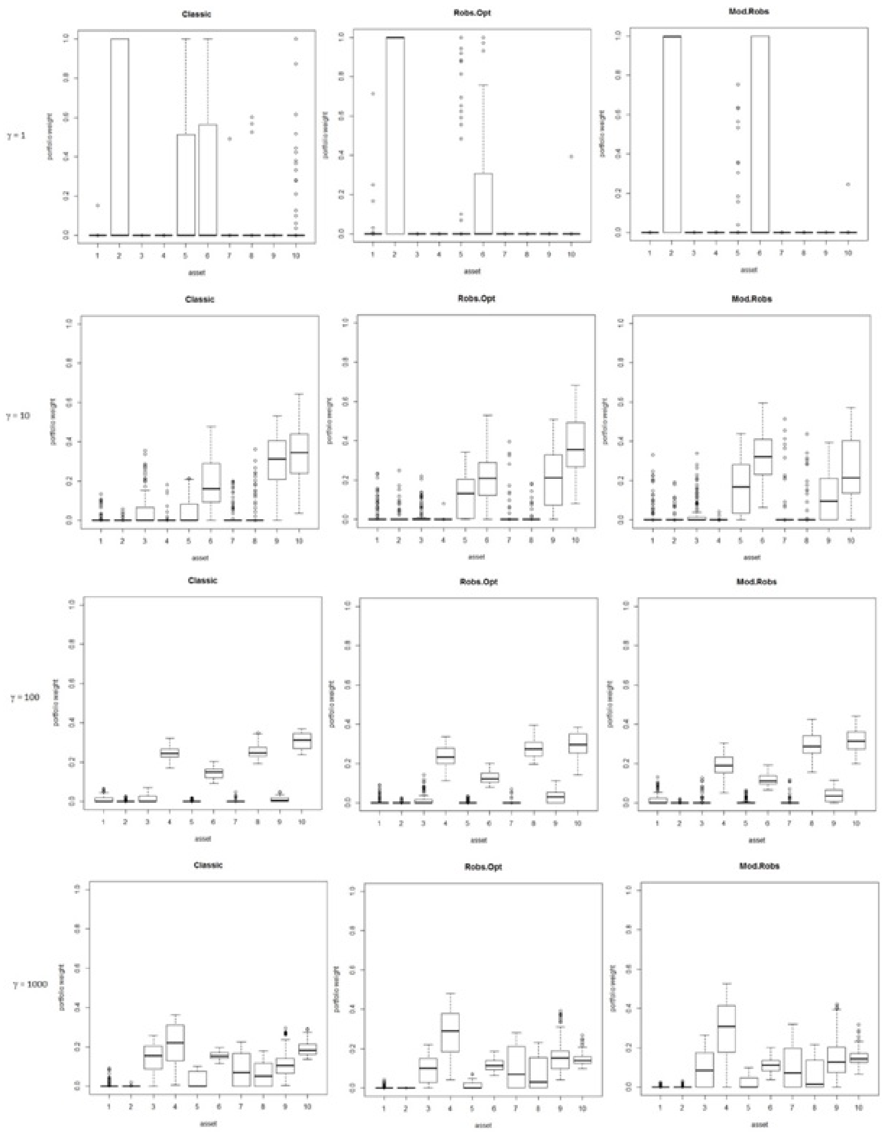

Figure 1: Boxplot of Portfolio Weights on Uncontaminated Data (no outlier)

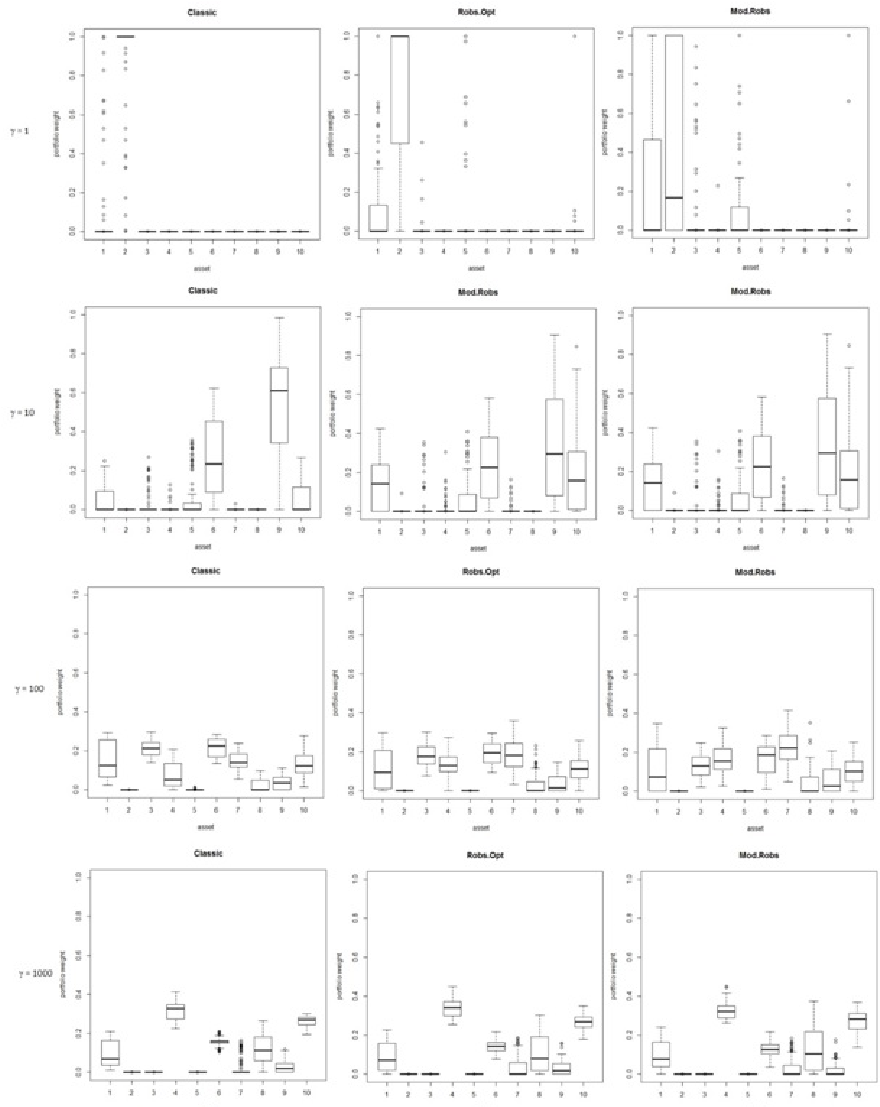

Figure 2: Boxplot of Portfolio Weights on Contaminated Data (outlier)

Finally, the simulated data set allows us to explore how different portfolios perform on those data when the asset returns deviate from normality. We have explored this issue on a simulated data set containing 5% of returns deviating from normality. This results are expected since classic portfolio and robust portfolio optimization method considered in this paper depend on the use of the same critical inputs i.e. the sample mean and the sample covariance matrix, which are MLE under the assumption of normality. Considering that the simulated data come from the contaminated data, the sample estimates are no longer MLE and will carry more estimation error.

To sum up, if the return data follow a distribution M that deviates slightly from the normal distribution, we could conclude that the Mod.Rob technique leads to an improvement compared to the Classic and Rob.Opt approaches. This improvement is possible due to the properties of robust estimators that can reduce outliers in data.

5. Empirical Analysis

The research utilizes historical weekly rates of return from ten companies from the Jakarta Stock Exchane (JSE), in January 2011 and Desember 2014 (207 observations). These rates of returns are presented in the scatterplots given in Figure 3. The ten companies are AALI (Astra Agro Lestari Tbk), ADHI (Adhi Karya Tbk), BBRI (Bank Rakyat Indonesia Tbk), BMRI (Bank Mandiri Tbk), CTRA, GGRM (Gudang Garam Tbk), ICBP (Indofood CBP Sukses Makmur Tbk), INDF(Indofood Sukses Makmur Tbk), INTP (Indocement Tunggal Prakasa Tbk) and MPPA (Matahari Putra Prima Tbk).

We noticed that some returns of the stocks have a few extreme returns (see Table 3). As stated by Sumiyana (2007) who studied the behavior of stock price variability in JSE, return variance in trading has differed significantly. These extreme returns heavily bias the sample mean vector and covariance matrix, and these outliers can lead to the assumption of normality of the data that is not fulfilled.

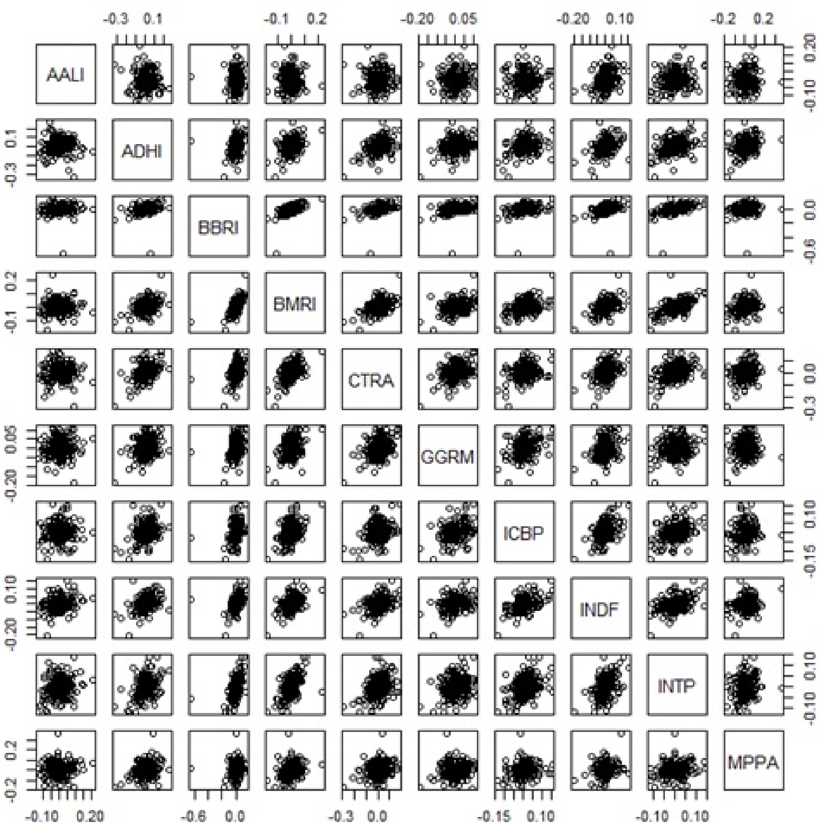

Figure 3: Scatter plot of logarithmic weekly returns of ten stocks

In this section, we analyse a real example where the portfolios are generated according to the "rolling-horizon" procedure similar to the method in the simulation study. Portfolio weights are estimated weekly using the last 90 weeks and rebalanced every week. In the sample there are 207 weekly returns, so there are 117 rebalances. On the basis of the estimated weights, portfolio returns are calculated - 117 out-of-sample weekly returns. Then, the resulting rates of return are used to estimate the mean, standard deviation, sharpe ratio and turnover.

Table 2: . Out-of-sample Mean, Standard Deviation, Sharpe Ratio and Turnover in Empirical Data

	Classic	Rob.Opt	Mod.Rob
$\gamma = 1$			
mean	0.0040	0.0082	0.0127
St.Dev	0.0444	0.0620	0.0506
SR	0.0894	0.1317	0.2513
TO	1.6686	1.6481	1.5151
$\gamma = 10$			
mean	0.0050	0.0062	0.0073
St.Dev	0.0378	0.0401	0.0327
SR	0.1318	0.1539	0.2217
TO	1.5585	1.5178	1.4427
$\gamma = 100$			
Mean	0.0040	0.0040	0.0043
St.Dev	0.0303	0.0317	0.0303
SR	0.1326	0.1274	0.1419
TO	1.2157	1.3486	1.1613
$\gamma = 1000$			
Mean	0.0039	0.0040	0.0041
St.dev	0.0311	0.0323	0.0308
SR	0.1248	0.1238	0.1331
TO	1.0779	1.2916	1.0906

Table (2) gives the out-of-sample performance for all the three portfolios in the empirical data. We saw that the mean and Sharpe ratio of the Mod.Rob portfolios are much larger than the Classic and Rob.Opt portfolios in all levels of risk aversion criteria. In addition, the out-of-sample risks (mesured by standard deviation) of our proposed model are lower compared to those of two portfolios.

Also, we observed that out-of-sample turnovers of the Mod.Rob are lower than the Classic and Mod.Rob in cases $\gamma = 1$ and 10. Meanwhile, when the risk aversion increases (i.e. $\gamma = 100$ and 1000) the performance of the out-of-sample turnover of Classic is better than that of Rob.Opt and Mod.Opt.

Also, the out-of-sample evaluation results show that the performances of the Rob.Opt portfolios proposed by Tütüncü and Koenig (2004) are not as good as those of our proposed robust portfolios, but they are better than those of the classic minimum-variance portfolios.

Next, we discussed the stability of the portfolio weights of different policies. Figure 4 gives the boxplots of the portfolio weights for all policies i.e. MV portfolio (Classic), robust portfolio optimization (Rob.Opt) and modified robust portfolio optimization (Mod.Rob). Each panel contains 10 boxplots corresponding to each of the ten assets. The ten assets are labeled as AALI = 1, ADHI = 2, BBRI = 3, BMRI = 4, CTRA = 5, GGRM = 6, ICBP = 7, INDF = 8, INTP = 9 and MPPA = 10. The boxplots for the portfolio weights for different values of γ are given per line that is risk aversion parameter $\gamma = 1$ (the first row of the graphs), $\gamma = 10$ (the second row of the graphs), $\gamma = 100$ (the third row of the graphs) and $\gamma = 1000$ (the fourth row of the graphs).

Figure (4) confirms the analysis of the portfolio weights on the simulated data, that the greater risk aversion will produce lower turnover value, meaning that portfolio is more stable.

Based on our observation on the empirical data (when there are outliers in the returns data), our out-of-sample evaluation results, however, show that our proposed models (Mod.Rob) have substantially outperformed the Classic and Rob.Opt portfolios. As argued previously, the reason for this is that the estimates of the parameters (both sample mean vector and matrix covariance) contain so many estimation errors that using them for portfolio selection is likely to hamper the performance of the resulting portfolios. Also, we saw among the robust approaches, the Mod.Rob approach clearly outperforms the Rob.Opt strategy. This improvement is possible due to the properties of robust estimators which are not influenced by the presence of outliers.

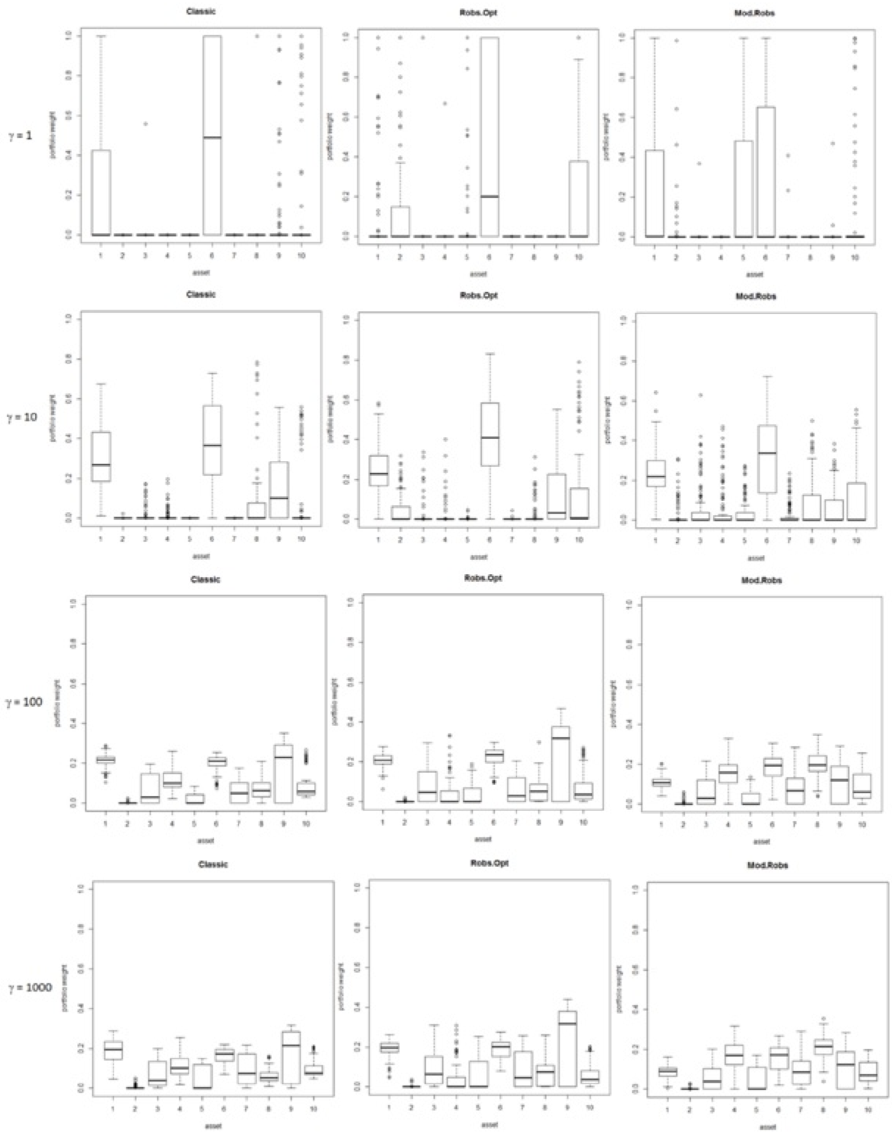

Figure 4: Boxplot of Portfolio Weight in Real Data Market

6. Conclusions

In this paper, we extended the robust portfolio optimization approach of Tütüncü and Koenig (2004) by incorporating robust estimators into a construction of the uncertainty sets of parameters. By doing this, our proposed model (Mod.Rob) is capable of tackling extreme data (outliers) in the uncertainty sets and, overall, leads to superior results over the Classic and Rob.Opt approaches both in the simulated and empirical data. We found that the out-of-sample portfolio risk of Mod.Rob is lower and accompanied by larger returns. Also, the portfolio compositions are more stable and consequently attain lower turnover compared to Classic and Rob.Opt approach.

The classic portfolio (MV) and original robust portfolio optimization approaches which use the same critical inputs i.e. the sample mean and the sample covariance matrix tend to be too pessimistic, especially when there are extreme returns in data set. Furthermore, the employed classical estimators (i.e. sample mean vector and covariance matrix) need to rely on distributional assumptions. Therefore both the Classic and Rob.Opt approaches exhibit the disadvantage of limited effectiveness when there are extreme observations (outliers) in data sets.

We found that the explicit incorporation of the robust estimators into the robust optimization process leads overall to superior results over the Classic and Rob.Opt approach, due to the properties of robust estimators which are not influenced by the presence of outliers.

References

Ben-Tal, A. and Nemirovski, A. (1998). Robust convex optimization. *Mathematics of Operations Research*, 23:769–805.

Ben-Tal, A. and Nemirovski, A. (2002). Robust optimization: methodology and applications. *Mathematical Programming. Ser B*, 92(3):453–480.

Bertsimas, D. and Sim, M. (2004). The price of robustness. *Operational Research*, 52(1):35–53.

Best, M. and Grauer, R. (1991). On the sensitivity of mean-variance efficient portfolios to changes in asset means: some analytical and computational results. *Review of Financial Studies*, 4:315–342.

Broadie, M. (1993). Computing effcient frontiers using estimated parameters. *Annals of Operations Research*, 45:21–58.

Ceria, S. and Stubbs, R. (2006). Incorporating estimation errors into portfolio selection: Robust portfolio construction. *Journal of Asset Management*, 7:109–127.

Chopra, V. and Ziemba, W. (1993). The effects of errors in means, variances, and covariances on optimal portfolio choice. *Journal of Portfolio Management*, 19:6–11.

Davies, P. L. (1987). Asymptotic behavior of s-estimatorss of multivariate location parameters and dispersion matrices. *Ann. Statist*, 15:1269–1292.

DeMiguel, V., Martin-Utrera, A., and Nogales, F. (2013). Size matters: Optimal calibration of shrinkage estimators for portfolio selection. *Journal of Banking & Finance*, 37:3018–3034.

Efron, B. and Tibshirani, R. (1993). *An Introduction to the Bootstrap*. Chapmann and Hall, New York.

Engels, M. (2004). *Extreme Value Modelling for Sports Data*. PhD thesis, Leiden Universiteit.

Fabozzi, J., Kolm, P., Pachamanova, D., and Focardi, F. (2007). *Robust Portfolio Optimization and Management*. John Wiley and Sons, New Jersey.

Fastrich, B. and Winker, P. (2009). Robust portfolio optimization wih a hybrid heuristik algorithm. *Computational Management Science*, 9(1):63–88.

Garlappi, L., Uppal, R., and Wang, T. (2004). Portfolio selection with parameter and model uncertainty: multi-prior approach. *Review of Financial Studies*, 20(1):41–81.

Goldfarb, D. and Iyengar, G. (2003). Robust portfolio selection problems. *Mathematics of Operations Research*, 28:1–38.

Huber, P. and Ronchetti, E. (2009). *Robust Statistics*. John Wiley and Sons, New York, 2nd edition.

Lauprete (2001). Disertation, Massachusetts Institue of Technology.

Lopuhaa, H. P. (2009). On the relation between s-estimatorss and m-estimatorss of multivariate location and covariance. *Ann. Statist*, 17:1662–1683.

Lu, Z. (2011). A computational study on robust portfolio selection based on a joint ellipsoidal uncertainty set. *Journal of Math. Program Ser A*, 126.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7:77–91.

Marona, R., Martin, R., and Yohai, V. (2006). *Robust Statistics:Theory and Methods*. John Wiley and Sons, New Jersey.

Rocke, D. (1996). Robustness properties of s-estimatorss of multivariate location and shape in high dimension. *The Annals of Statistics*, 24(3):1327–1345.

Sumiyana (2007). Behavior of stock price variability over trading anf non-trading, and daily return volatility. *Gadjah Mada International Journal of Business*, 9(3):409–453.

Supandi, E., Rosadi, D., and Abdurakhman (2016). An empirical comparison between robust estimation and robust optimization to mean-variance portfolio. *Journal of Modern Applied Statistical Methods. in press.*

Tütüncü, R. and Koenig, M. (2004). Robust asset allocation. *Annals of Operations Research*, 13:157–187.

Zhu, L. (2008). *Optimal Portfolio Selection Under the Estimation Risk in mean Return*. PhD thesis, University of Waterloo, Canada.